Tutorial 6 (3 Mar)

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Short Instructions of submission of Midterm papers

Date and Time 6 Mar (Sat) ; 10:30 – 12:00

Format · Students will be divided to 3 groups

- There will be 3 Midterm papers, one for each group.
- Students will receive their midterm exam paper via CUHK emails.
- Students are required to finish the received paper only.
- After finishing the paper, students should scan and submit the solutions to the corresponding item in Gradescope

Flow

Period	Content
10:25 – 10:30	Distributing Midterm papers via group emails
10:30 – 12:00	Midterm Examination
12:00 – 12:20	Submission Period
12:20 – 23:00	Late Submission Period

Rmk · Students MUST match their solutions with the outline in Gradescope. Otherwise, 5 marks (out of 100) will be deducted.

- Further details could be found in the "Midterm Guideline".

Change of Variables Formula

Thm (Change of Variables Formula) Given the following datum:

- $D_1, D \subseteq \mathbb{R}^n$ be two regions in $\mathbb{R}^n$.
- $\Phi = (\Phi_1, \cdots, \Phi_n) : D_1 \longrightarrow D$ be a C^1-diffeomorphism

$$\begin{array}{ccc} D_1 & \longrightarrow & D \\ \Psi & & \Psi \\ (y_1, \cdots, y_n) = \vec{y} & \longmapsto & \Phi(\vec{y}) = (\Phi_1(y_1, \cdots, y_n), \cdots, \Phi_n(y_1, \cdots, y_n)) \end{array}$$

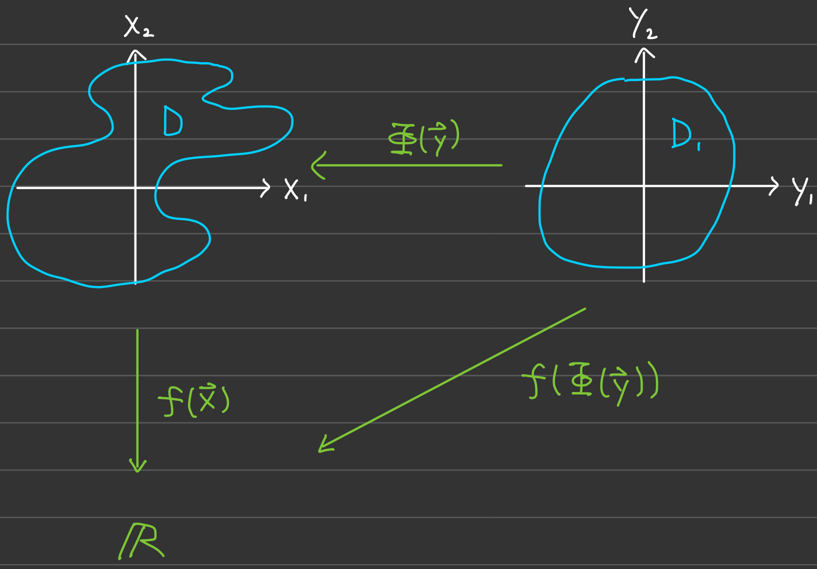
- $J_\Phi := \det(\nabla \Phi) : D_1 \longrightarrow \mathbb{R}$ be its Jacobian.

$$\begin{array}{ccc} D_1 & \longrightarrow & \mathbb{R} \\ \Psi & & \Psi \\ \vec{y} & \longmapsto & J_\Phi(\vec{y}) = \det(\nabla \Phi(\vec{y})) = \det\left(\frac{\partial \Phi_i}{\partial y_j}(\vec{y})\right) \end{array}$$

then for any continuous function $f : D \longrightarrow \mathbb{R}$,

$$\int_D f(\vec{x})\, d\vec{x} = \int_{D_1} f(\Phi(\vec{y}))\, |J_\Phi(\vec{y})|\, d\vec{y} \qquad -(*)$$

2-dimensional Picture



Revision via Change of Variables Formula

Cor For $n=2$, replacing (x_1, x_2) by (x,y) and (y_1, y_2) by (u,v),

$$(*) \Rightarrow \iint_D f(x,y)\,dA(x,y) = \iint_{D_1} f(\Phi(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v)$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = J_\Phi(u,v) = \begin{vmatrix} \frac{\partial \Phi_1}{\partial u}(u,v) & \frac{\partial \Phi_1}{\partial v}(u,v) \\ \frac{\partial \Phi_2}{\partial u}(u,v) & \frac{\partial \Phi_2}{\partial v}(u,v) \end{vmatrix} =: \begin{vmatrix} \frac{\partial x}{\partial u}(u,v) & \frac{\partial x}{\partial v}(u,v) \\ \frac{\partial y}{\partial u}(u,v) & \frac{\partial y}{\partial v}(u,v) \end{vmatrix}$$

Cor (Polar coordinate system in $\mathbb{R}^2$)

Define

$$\begin{array}{ccc} \Phi : D_1 \subseteq [0,+\infty) \times [0,2\pi) & \longrightarrow & D \subseteq \mathbb{R}^2 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (r,\theta) & \longmapsto & (r\cos\theta, r\sin\theta) \end{array}$$

then $\frac{\partial(x,y)}{\partial(r,\theta)} = r \geq 0$

$$\Rightarrow \iint_D f(x,y)\,dA(x,y) = \iint_{D_1} f(r\cos\theta, r\sin\theta)\, r \cdot dA(r,\theta)$$

$$= \int_{\theta_1}^{\theta_2} \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta. \quad \text{(by Fubini's Thm)}$$

if $D_1 = \{ (r,\theta) \in [0,+\infty) \times [0,2\pi) \mid \underbrace{\theta_1 \leq \theta \leq \theta_2}_{\text{(If } \theta_1 = 0;\ \theta_2 = 2\pi\text{, replaced by } 0 \leq \theta < 2\pi)} ;\ \varphi_1(\theta) \leq r \leq \varphi_2(\theta) \}$, where

- $\theta_1, \theta_2 \in [0,2\pi]$ are constants satisfying $\theta_1 < \theta_2$.
- $\varphi_1, \varphi_2 : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ are continuous satisfying $0 \leq \varphi_1(\theta) \leq \varphi_2(\theta)$ for any $\theta \in [\theta_1, \theta_2]$

Cor For $n=3$, replacing D, D_1 by Ω, Ω_1, (x_1, x_2, x_3) by (x,y,z) and (y_1, y_2, y_3) by (u,v,w), $\frac{\partial(x,y,z)}{\partial(u,v,w)} := J_{\Phi}(u,v,w)$

$$(*) \Rightarrow \iiint_{\Omega} f(x,y,z)\,dV(x,y,z) = \iiint_{\Omega_1} f(\Phi(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV(u,v,w)$$

Cor ① (Cylindrical coordinate system in $\mathbb{R}^3$)

Define

$$\begin{array}{ccc} \Phi : \Omega_1 \subseteq [0,+\infty) \times [0,2\pi) \times \mathbb{R} & \longrightarrow & \Omega \subseteq \mathbb{R}^3 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (r,\theta,z) & \longmapsto & (r\cos\theta, r\sin\theta, z) \end{array}$$

then $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r \geq 0$

$$\Rightarrow \iiint_{\Omega} f(x,y,z)\,dV(x,y,z) = \iiint_{\Omega_1} f(r\cos\theta, r\sin\theta, z)\, r\, dV(r,\theta,z)$$

$$= \int_{\theta_1}^{\theta_2} \int_{h_1(\theta)}^{h_2(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta \quad \text{(by Fubini's Thm)}$$

if $\Omega_1 := \{(r,\theta,z) \mid (r,\theta) \in R,\ f_1(r,\theta) \leq z \leq f_2(r,\theta)\}$

$$= \{(r,\theta,z) \mid \underbrace{\theta_1 \leq \theta \leq \theta_2}_{};\ h_1(\theta) \leq r \leq h_2(\theta),\ f_1(r,\theta) \leq z \leq f_2(r,\theta)\}, \text{ where}$$

(If $\theta_1 = 0$; $\theta_2 = 2\pi$, replaced by $0 \leq \theta < 2\pi$)

- $\theta_1, \theta_2 \in [0,2\pi]$ are constants satisfying $\theta_1 < \theta_2$.
- $h_1, h_2 : [\theta_1, \theta_2] \to \mathbb{R}$ are continuous satisfying $0 \leq h_1(\theta) \leq h_2(\theta)$ for any $\theta \in [\theta_1, \theta_2]$
- $f_1, f_2 : R \to \mathbb{R}$ are continuous with $f_1(r,\theta) \leq f_2(r,\theta)$, $\forall (r,\theta) \in R$.

Cor ② Spherical coordinate system in $\mathbb{R}^3$

Define $\Phi : \Omega_1 \subseteq [0,+\infty) \times [0,\pi] \times [0,2\pi) \longrightarrow \Omega \subseteq \mathbb{R}^3$

$$\begin{array}{ccc} \Omega_1 & \longrightarrow & \Omega \\ \cup & & \cup \\ (\rho,\varphi,\theta) & \longmapsto & (\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi) \end{array}$$

then $\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \rho^2\sin\varphi \geq 0$

$$\Rightarrow \iiint_{\Omega} f(x,y,z)\,dV(x,y,z) = \iiint_{\Omega_1} f(\Phi(\rho,\varphi,\theta))\,\rho^2\sin\varphi\,dV(\rho,\varphi,\theta)$$

$$= \int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} f(\Phi(\rho,\varphi,\theta))\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

if $\Omega_1 = \{(\rho,\varphi,\theta) \mid \underbrace{\theta_1 \leq \theta \leq \theta_2}_{}, \varphi_1 \leq \varphi \leq \varphi_2, \rho_1(\varphi,\theta) \leq \rho \leq \rho_2(\varphi,\theta)\}$

(If $\theta_1 = 0$; $\theta_2 = 2\pi$, replaced by $0 \leq \theta < 2\pi$)

- $\theta_1, \theta_2 \in [0,2\pi]$ are constants satisfying $\theta_1 < \theta_2$.
- $\varphi_1, \varphi_2 \in [0,\pi]$ are constants satisfying $\varphi_1 < \varphi_2$.
- $\rho_1, \rho_2 : [\varphi_1,\varphi_2] \times [\theta_1,\theta_2] \to \mathbb{R}$ are continuous with $0 \leq \rho_1(\varphi,\theta) \leq \rho_2(\varphi,\theta)$,

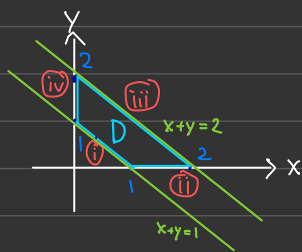
$\forall (\varphi,\theta) \in [\varphi_1,\varphi_2] \times [\theta_1,\theta_2]$

Ex Evaluate $\iint_D \cos\frac{y-x}{y+x}\,dA$, where D is the trapezoidal region with vertices $(1,0), (2,0), (0,2), (0,1)$.

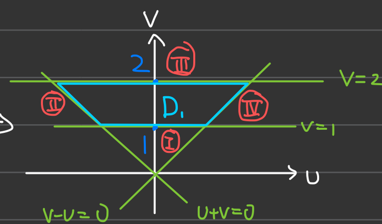
Sol Idea: Simplify the integrand by a change of variables and compute the integral.

Step 1 Apply a change of variables

$$\begin{cases} y-x = u \\ y+x = v \end{cases} \Longleftrightarrow \begin{cases} x = \frac{v-u}{2} \\ y = \frac{v+u}{2} \end{cases}$$



$\Longleftrightarrow$



Boundary equations

$$\left\{\begin{array}{l} (i)\ x+y=1 \\ (ii)\ y=0 \\ (iii)\ x+y=2 \\ (iv)\ x=0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} (I)\ v=1 \\ (II)\ u+v=0 \\ (III)\ v=2 \\ (IV)\ v-u=0 \end{array}\right\}$$

$\therefore D_1 = \{(u,v)\in\mathbb{R}^2 \mid 1 \le v \le 2,\ -v \le u \le v\}$

Step 2 Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Step 3 Apply the change of variables formula.

$$\iint_D \cos\left(\frac{y-x}{y+x}\right)dx\,dy = \iint_{D_1} \cos\frac{u}{v}\cdot\left(\left|-\tfrac{1}{2}\right|du\,dv\right) = \frac{1}{2}\int_1^2\int_{-v}^{v}\cos\frac{u}{v}\,du\,dv = \frac{1}{2}\int_1^2\left[v\sin\frac{u}{v}\right]_{-v}^{v}dv$$

$$= \frac{1}{2}\int_1^2 \left(v\sin 1 - (-v\sin 1)\right)dv = (\sin 1)\cdot\int_1^2 v\,dv = \sin 1\cdot\left[\frac{v^2}{2}\right]_1^2 = \frac{3}{2}\sin 1$$